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# INCLUSION RESULTS ASSOCIATED WITH CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING CALCULUS OPERATOR

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Abstract. In this paper, a generalized class of starlike functions involving calculus operator is defined and a coefficient inequality is applied to find certain mappings of the normalized calculus operator $\widetilde{\mathcal{I}}_{\nu}^{\delta}$ under some parametric restrictions.

Keywords: Gaussian hypergeometric functions, convex functions, starlike functions, uniformly starlike functions, uniformly convex functions, Hadamard product, Carlson-Shaffer operator, calculus operator.

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## 1. Introduction

Let $\mathcal{A}$ be the class of analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are analytic, univalent in $\mathbb{U}$ and normalized by $f(0) = 0 = f'(0) - 1$. Also denote by $\mathcal{T}$ the class of analytic functions with negative coefficients (introduced by Silverman [28]) consisting of functions $f$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{U} \tag{2}$$

The class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha < 1$

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \ \Re \frac{zf'(z)}{f(z)} > \alpha, \ z \in \mathbb{U} \right\}.$$

and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha < 1$

$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{A} : \ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \ z \in \mathbb{U} \right\} = \left\{ f \in \mathcal{A} : \ zf' \in \mathcal{S}^*(\alpha) \right\}$$

were introduced by Robertson in [26]. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. Further, $\mathcal{K}(0) =: \mathcal{K}$ is the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \iff zf' \in \mathcal{S}^*(\alpha)$.

In 1993, Goodman [5, 6] introduced the concept of uniform convexity and uniform starlikeness for functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{U}$ if $f$ is a normalized

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convex univalent function and has the property that for every circular arc $\delta$ contained in the open unit disc $\mathbb{U}$, with centre $\zeta$ also in $\mathbb{U}$, the image curve $f(\delta)$ is a convex arc. Ronning [23] introduced the class $\mathcal{S}_P$, geometrically $\mathcal{S}_P$ is the class of functions $F$ for which $\frac{zF'(z)}{F(z)}$ has values in the interior of the parabola in the right half-plane symmetric about the real axis with vertex at $(1/2, 0)$. In [10], the geometric definition of $k$-$\mathcal{UCV}$ and its connections with the conic domains were considered. The class $k$-$\mathcal{ST}$ and its properties were investigated in [12]. The analytic characterizations of $k$-$\mathcal{UCV}$ and $k$-$\mathcal{ST}$ are as follows:

$$k-\mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathbb{U}) \right\} \text{ and}$$

$$k-\mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathbb{U}) \right\}.$$

Further, Kanas and Srivastava [11] presented a systematic and unified study of the classes $\mathcal{UCV}$ and $\mathcal{S}_P$. (also see [31]).

Let $\Omega_k$ be a domain such that $1 \in \Omega_k$ and

$$\partial\Omega_k = \{\omega = u + iv : u^2 = k^2(u-1)^2 + k^2v^2, u > 0\}, \quad 0 \le k < \infty.$$

The domain $\Omega_k$ elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic when $k = 1$ and a right half-plane when $k = 0$. If $p$ is an analytic function with $p(0) = 1$ which maps the unit disc $\mathbb{U}$ conformally onto the region $\Omega_k$, then $p'(0) = P_1(k)$ and

$$P_1(k) = \begin{cases} \frac{2A^2}{1-k^2} & \text{for } 0 < k < 1 \\ \frac{8}{\pi^2} & \text{for } k = 1 \\ \frac{\pi^2}{4\sqrt{t}(1+t)(k^2-1)K^2(t)} & \text{for } k > 1 \end{cases} \tag{3}$$

where $A = \frac{2}{\pi}\arccos k$ and $t \in (0,1)$ is determined by $k = \cosh(\pi\mathcal{K}'(t)/[4\mathcal{K}(t)])$, $\mathcal{K}(t)$ is the Legendre's complete Elliptic integral of the first kind

$$\mathcal{K}(t) = \int_0^1 \frac{d\chi}{\sqrt{(1-\chi^2)(1-t^2\chi^2)}}$$

and $\mathcal{K}'(t) = \mathcal{K}(\sqrt{1-t^2})$ is the complementary integral of $\mathcal{K}(t)$.
The concrete form of $P_1$ was given in [8, 9, 16, 30]. Further $P_1(k)$ is strictly decreasing function of the variable $k$ and its values are included in the interval (0, 2].

Let $f \in \mathcal{A}$ be of the form (1). If $f \in k$-$\mathcal{UCV}$, then the following coefficient inequalities hold true [10]:

$$|a_n| \le \frac{(P_1(k))_{n-1}}{(1)_n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{4}$$

Similarly, if $f \in \mathcal{A}$ be of the form (1) belongs to the class $k$-$\mathcal{ST}$, then [12]:

$$|a_n| \le \frac{(P_1(k))_{n-1}}{(1)_{n-1}}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{5}$$

The study of operators plays an important role in the geometric function theory and its related fields. We have seen many differential and integral operators defined soundly by convolution of certain analytic functions. This somehow helps us to understand better the geometric properties of such operators. For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\Omega$ be the class of functions $w$ which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \ (z \in \mathbb{U}).$$

Let $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$ then the function $p(z)$ is said to subordinate to $q(z)$ in $\mathbb{U}$ written by

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}), \tag{6}$$

such that $p(z) = q(w(z))$ $(z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (6) implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{7}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then the subordination (6) is equivalent to the condition (7).

We recall, for $\nu > -1; \delta \in \mathbb{R}$ the calculus operator $\mathcal{I}_\nu^\delta$ which was recently studied by Kim and Srivastava [14](see also [27]) and the image of $z^n$ under this operator is given by

$$\mathcal{I}_\nu^\delta z^n = \frac{\Gamma(\nu+1+n)}{\Gamma(\nu+1+\delta+n)} z^{n+\delta+\nu}$$

for positive $\nu + 1 + n > -\delta; (\delta \in \mathbb{R})$. Further, for positive $\nu + 2 > -\delta$; $(\delta \in \mathbb{R})$ and $f$ of the form (1),then the normalized operator $\widetilde{\mathcal{I}_\nu^\delta} f(z)$ of $\mathcal{I}_\nu^\delta f(z)$ is given by

$$\widetilde{\mathcal{I}}_\nu^\delta f(z) \quad = \quad \frac{\Gamma(\nu+2+\delta)}{\Gamma(\nu+2)} z^{-\delta-\nu} \mathcal{I}_\nu^\delta f(z) = z + \sum_{n=2}^{\infty} \Theta_n(\delta,\nu) a_n z^n \tag{8}$$

where

$$\Theta_n(\delta,\nu) = \frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} \tag{9}$$

and $(a)_n$ is the Pochhammer symbol given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\dots(a+n-1) \text{ and } (a)_0 = 1.$$

Note that for $\nu > -2$ and different choices of $\delta$, we get

$$\widetilde{\mathcal{I}}_\nu^0 f(z) \equiv f(z) \quad \text{and} \quad \widetilde{\mathcal{I}}_0^{-1} f(z) \equiv z f'(z).$$

Further, for $\lambda > -1$; $\nu = \lambda$ and $\delta = -\lambda$, we have

$$\widetilde{\mathcal{I}}_{-\lambda}^{\lambda} f(z) \equiv \mathcal{D}^\lambda f(z)$$

the Ruscheweyh derivative operator [25] and

$$\widetilde{\mathcal{I}}_{a-2}^{c-a} f(z) \equiv \mathcal{L}(a,c)$$

is the Carlson-Shaffer operator [3].

In this paper, due to Ramesha et al. [22], Obradovic and Joshi [19], Padmanabhan [21], Nunokawa et al. [18], we introduce a new subclass $\mathcal{G}_\nu^\delta(\lambda,\beta)$ of $\mathcal{A}$ involving calculus operator to obtain coefficient estimate and obtain maximization of $|a_3 - \mu a_2^2|$. Further we discussed certain mappings to the class $\mathcal{G}_\nu^\delta(\lambda,\beta)$ of the operator $\widetilde{\mathcal{I}}_\nu^\delta$ if some parametric inequalities hold.

For positive $\nu + 2 > -\delta, (\delta \in \mathbb{R})$; $0 \le \lambda < 1$ and $0 \le \beta < 1$, we let a generalized class $\mathcal{G}_\nu^\delta(\lambda,\beta)$ the subclass of functions $f(z) \in \mathcal{A}$ which satisfy the condition

$$\Re\left(\frac{z(\widetilde{\mathcal{I}}_\nu^\delta f(z))' + \lambda z^2 (\widetilde{\mathcal{I}}_\nu^\delta f(z))''}{\widetilde{\mathcal{I}}_\nu^\delta f(z)}\right) > \beta, \ (z \in \mathbb{U}). \tag{10}$$

Also denote $\mathcal{TG}_\nu^\delta(\lambda,\beta) = \mathcal{G}_\nu^\delta(\lambda,\beta) \cap \mathcal{T}$. Equivalently a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{G}_\nu^\delta(\lambda,\beta)$ if and only if

$$\frac{z(\widetilde{\mathcal{I}}_\nu^\delta f(z))' + \lambda z^2(\widetilde{\mathcal{I}}_\nu^\delta f(z))''}{\widetilde{\mathcal{I}}_\nu^\delta f(z)} \prec \frac{1+(1-2\beta)w(z)}{1-w(z)} \tag{11}$$

where $w(z) \in \Omega$.

**Remark 1.1.** *It is of interest to note that for $\lambda = 0$, we have $\mathcal{G}_\nu^\delta(\lambda,\beta) \equiv \mathcal{S}_\nu^\delta(\beta)$*

We obtain the following necessary and sufficient conditions for functions $f \in \mathcal{G}_\nu^\delta(\lambda,\beta)$.

## 2. Coefficient estimate

**Theorem 2.1.** *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{G}_\nu^\delta(\lambda,\beta)$ if*

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\ \Theta_n(\delta,\nu)|a_n| \le 1-\beta. \tag{12}$$

*Proof.* Since $0 \le \beta < 1$ and $\lambda \ge 0$, now for the function

$$P(z) = \frac{z(\widetilde{\mathcal{I}}_\nu^\delta f(z))' + \lambda z^2(\widetilde{\mathcal{I}}_\nu^\delta f(z))''}{\widetilde{\mathcal{I}}_\nu^\delta f(z)}.$$

We prove that $|P(z)-1| < 1-\beta, (z \in \mathbb{U})$. Indeed if $f(z) \equiv z(z \in \mathbb{U})$, then we have $P(z) \equiv 1(z \in \mathbb{U})$. This implies that the desired in equality (12). If $f(z) \neq z(z \in \mathbb{U})$, then there exist a coefficient $\Theta_n(\delta,\nu)a_n \neq 0$ for some $n \ge 2$. It follows that $\sum\limits_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n| > 0$. Further note that

$$\sum_{n=2}^{\infty}\left[\lambda n^2+n-\lambda n-\beta\right]\Theta_n(\delta,\nu)|a_n| > (1-\beta)\sum_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n|$$

which implies that

$$\sum_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n| < 1.$$

By coefficient inequality (12), we thus obtain

$$\begin{aligned}
|P(z)-1| &= \left|\frac{\sum\limits_{n=2}^{\infty}(n-1)(n\lambda+1)\Theta_n(\delta,\nu)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\Theta_n(\delta,\nu)a_nz^{n-1}}\right| \\
&\le \frac{\sum\limits_{n=2}^{\infty}(n-1)(n\lambda+1)\Theta_n(\delta,\nu)|a_n|}{1-\sum\limits_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n|} \\
&\le \frac{\sum\limits_{n=2}^{\infty}\left[\lambda n^2+n-\lambda n-\beta\right]\Theta_n(\delta,\nu)|a_n| - (1-\beta)\sum\limits_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n|}{1-\sum\limits_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n|} \\
&\le \frac{(1-\beta)-(1-\beta)\sum\limits_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n|}{1-\sum\limits_{n=2}^{\infty}\Theta_n(\delta,\nu)|a_n|}, (z \in \mathbb{U}).
\end{aligned}$$

Hence we obtain

$$\Re\left(\frac{z(\widetilde{\mathcal{I}}_\nu^\delta f(z))' + \lambda z^2(\widetilde{\mathcal{I}}_\nu^\delta f(z))''}{\widetilde{\mathcal{I}}_\nu^\delta f(z)}\right) = \Re(P(z)) > 1 - (1-\beta) = \beta, (z \in \mathbb{U}).$$

That is $f \in \mathcal{G}_\nu^\delta(\lambda, \beta)$. This completes the proof. □

In the next theorem, we show that the condition (12) is also necessary for functions $f \in \mathcal{TG}_\nu^\delta(\lambda, \beta)$.

**Theorem 2.2.** *Let $f$ be given by (2) then the function $f \in \mathcal{TG}_\nu^\delta(\lambda, \beta)$ if and only if (12) holds.*

*Proof.* In view of Theorem 2.1 we need only to show that $f \in \mathcal{TG}_\nu^\delta(\lambda, \beta)$ satisfies the coefficient inequality (12). If $f \in \mathcal{TG}_\nu^\delta(\lambda, \beta)$ then by definition, we have conversely assume that (12) holds. Let

$$P(z) = \frac{z(\widetilde{\mathcal{I}}_\nu^\delta f(z))' + \lambda z^2(\widetilde{\mathcal{I}}_\nu^\delta f(z))''}{\widetilde{\mathcal{I}}_\nu^\delta f(z)}.$$

Then we have $\Re(P(z)) > \beta$ this implies that

$$\widetilde{\mathcal{I}}_\nu^\delta f(z) = z - \sum_{n=2}^{\infty} \Theta_n(\delta, \nu) a_n z^n \neq 0; (z \in \mathbb{U} \setminus \{0\}).$$

Noting that $\frac{\widetilde{\mathcal{I}}_\nu^\delta f(r)}{r}$ is the real continuous function in the open interval $(0,1)$ with $f(0) = 1$, we have

$$1 - \sum_{n=2}^{\infty} \Theta_n(\delta, \nu) a_n r^{n-1} > 0, (0 < r < 1). \tag{13}$$

Now

$$\beta < P(r) = \frac{1 - \sum_{n=2}^{\infty}[\lambda n^2 + n - \lambda n]\Theta_n(\delta, \nu) a_n r^{n-1}}{1 - \sum_{n=2}^{\infty} \Theta_n(\delta, \nu) a_n r^{n-1}}$$

and consequently by (13) we obtain

$$\sum_{n=2}^{\infty}[\lambda n^2 + n - \lambda n - \beta]\Theta_n(\delta, \nu) a_n < 1 - \beta.$$

Letting $r \to 1$, and using (12) we get

$$|P(z) - 1| \leq 1 - \beta$$

which implies that

$$\Re\left(P(z)\right) > \beta.$$

This proves the converse part. □

**Corollary 2.1.** *(Coefficient Estimate) If a function $f$ of the form (2) belongs to the class $\mathcal{TG}_\nu^\delta(\lambda, \beta)$, then*

$$|a_n| \leq \frac{1 - \beta}{[\lambda n^2 + n - \lambda n - \gamma]\,\Theta_n(\delta, \nu)} \quad n = 2, 3, \ldots.$$

*The equality holds for the functions*

$$h_n(z) = z - \frac{1 - \beta}{[\lambda n^2 + n - \lambda n - \beta]\,\Theta_n(\delta, \nu)} z^n, \quad z \in \mathbb{U},\ n = 2, 3, \ldots. \tag{14}$$

**Remark 2.1.** *Making use of the necessary and sufficient conditions and the coefficient estimate one can easily obtain the distortion bounds,extreme points, integral means and neighbourhood results,convolution and inclusion results for functions $f \in \mathcal{TG}_{\nu}^{\delta}(\lambda, \beta)$ proceeding as in the work of Dziok and Murugusundaramoorthy in* [4]*(see also the references cited therein) and Silverman* [28].

## 3. Maximization of $|a_3 - \mu a_2^2|$

**Lemma 3.1.** [13] *Let $w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$. If $\mu$ is any complex number, then*

$$|c_2 - \mu c_1^2| \leq max\{1, |\mu|\}.$$

*For convenience we let $\Theta_n = \Theta_n(\delta, \nu)$ for our study.*

**Theorem 3.1.** *If a function $f(z)$ defined by (1) is in the class $\mathcal{G}_{\nu}^{\delta}(\lambda, \beta)$ and $\mu$ is any complex number then*

$$|a_3 - \mu a_2^2| \leq \max\{1, |d|\},$$

*where*

$$d = \frac{2(1-\beta)(6\lambda+2)\Theta_3 - (2\lambda+3-2\beta)(2\lambda+1)\Theta_2^2}{(2\lambda+1)^2\Theta_2^2}.$$

*Proof.* Since $f \in \mathcal{G}_{\nu}^{\delta}(\lambda, \beta)$, by (11) we have

$$\frac{z(\widetilde{\mathcal{I}}_{\nu}^{\delta} f(z))' + \lambda z^2 (\widetilde{\mathcal{I}}_{\nu}^{\delta} f(z))''}{\widetilde{\mathcal{I}}_{\nu}^{\delta} f(z)} \prec \frac{1 + (1-2\beta)w(z)}{1 - w(z)}$$

by simple computation we get

$$\begin{aligned} w(z) &= \frac{\lambda z^2 \widetilde{\mathcal{I}}(f)'' + z\widetilde{\mathcal{I}}(f)' - \widetilde{\mathcal{I}}(f)}{\lambda z^2 \widetilde{\mathcal{I}}(f)'' + z\widetilde{\mathcal{I}}(f)' + (1-2\beta)\widetilde{\mathcal{I}}(f)} \\ &= \frac{\sum_{n=2}^{\infty}[n(n-1)\lambda + (n-1)]a_n\Theta_n z^n}{2(1-\beta)z + \sum_{n=2}^{\infty}[n(n-1)\lambda + (n+1-2\beta)]a_n\Theta_n z^n} \\ &= \frac{\sum_{n=2}^{\infty}[n(n-1)\lambda + (n-1)]a_n\Theta_n z^{n-1}}{2(1-\beta)} \times \left(1 + \frac{\sum_{n=2}^{\infty}[n(n-1)\lambda + (n+1-2\beta)]a_n\Theta_n z^{n-1}}{2(1-\beta)}\right)^{-1}. \end{aligned}$$

Thus,

$$w(z)\left(1 + \frac{\sum_{n=2}^{\infty}[n(n-1)\lambda + (n+1-2\beta)]a_n\Theta_n z^{n-1}}{2(1-\beta)}\right) = \frac{\sum_{n=2}^{\infty}[n(n-1)\lambda + (n-1)]a_n\Theta_n z^{n-1}}{2(1-\beta)}. \tag{15}$$

By comparing the coefficients of $z$ and $z^2$ in the above equation, we get,

$$a_2 = \frac{2(1-\beta)c_1}{(2\lambda+1)\Theta_2}, \quad \text{where} \quad \Theta_2 = \frac{\nu+2}{\nu+2+\delta} \tag{16}$$

and

$$a_3 = \frac{2(1-\beta)}{(6\lambda+2)\Theta_3}\left(c_2 + \frac{2\lambda+2-2\beta}{2\lambda+1}c_1^2\right), \quad \Theta_3 = \frac{(\nu+2)(\nu+3)}{(\nu+2+\delta)(\nu+3+\delta)}. \tag{17}$$

Hence,

$$a_3 - \mu a_2^2 = \frac{2(1-\beta)}{(6\lambda+2)\Theta_3}(c_2 - dc_1^2), \tag{18}$$

where

$$d = \frac{2(1-\beta)(6\lambda+2)\Theta_3 - (2\lambda+3-2\beta)(2\lambda+1)\Theta_2^2}{(2\lambda+1)^2\Theta_2^2}.$$

Taking modulus on both sides in (18),we have

$$|a_3 - \mu a_2^2| = \frac{2(1-\beta)}{(6\lambda+2)\Theta_3}|c_2 - dc_1^2|, \tag{19}$$

Using Lemma 3.1, we have

$$|a_3 - \mu a_2^2| = \frac{2(1-\beta)}{(6\lambda+2)\Theta_3} \max\{1, |d|\}.$$

□

## 4. Applications of calculus operator

The Theory of Special Functions play an important rôle in Geometric Function Theory, especially in the solution by de Branges [2] of the famous Bieberbach conjecture. There is an extensive literature dealing with geometric properties of different families of special functions, particularly the generalized hypergeometric functions [3, 7, 15, 20, 24, 29] .The Gaussian hypergeometric function $F(a, b; c; z)$ given by

$${}_2F_1(a, b; c; z) = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathbb{U})$$

near $z = 1$ is classified into three cases according as $\Re(c - a - b)$ is positive, zero or negative, respectively was studied by many authors on different counts.

Let

$$\widetilde{\mathcal{I}}_\nu^\delta u(z) = \frac{\Gamma(\nu+2+\delta)}{\Gamma(\nu+2)} z^{-\delta-\nu} \mathcal{I}_\nu^\delta u(z) = z + \sum_{n=2}^{\infty} \frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} z^n.$$

By using the Gauss Summation theorem [1]

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } \Re(c-a-b) > 0, \tag{20}$$

to $\widetilde{\mathcal{I}}_\nu^\delta u(z)$ and taking $z = 1$, simple computation yields

$$\widetilde{\mathcal{I}}_\nu^\delta u(1) = \sum_{n=2}^{\infty} \frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} = \frac{\Gamma(\nu+2+\delta)\Gamma(\delta-1)}{\Gamma(\delta)\Gamma(\nu+1+\delta)} - 1, \quad \delta > 1. \tag{21}$$

Differentiating $\widetilde{\mathcal{I}}_\nu^\delta u(z)$ with respect to $z$ and taking $z = 1$, we have

$$\sum_{n=2}^{\infty} \frac{n(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} = \frac{\Gamma(\nu+2+\delta)\Gamma(\delta-2)}{\Gamma(\delta)\Gamma(\nu+\delta)} - 1, \quad \delta > 2. \tag{22}$$

Again differentiating $(\widetilde{\mathcal{I}}_\nu^\delta u(z))'$ with respect to $z$ and taking $z = 1$, we have

$$\sum_{n=2}^{\infty} \frac{n(n-1)(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} = \frac{2(\nu+2)\Gamma(\nu+2+\delta)\Gamma(\delta-3)}{\Gamma(\delta)\Gamma(\nu+\delta)}, \quad \delta > 3. \tag{23}$$

Further differentiating $(\widetilde{\mathcal{I}}_\nu^\delta u(z))''$ with respect to $z$ and taking $z = 1$, we have

$$\sum_{n=2}^{\infty} \frac{n(n-1)(n-2)(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} = \frac{6(\nu+2)(\nu+3)\Gamma(\nu+2+\delta)\Gamma(\delta-4)}{\Gamma(\delta)\Gamma(\nu+\delta)}, \ \delta > 4. \tag{24}$$

Motivated by the works of Srivastava et al. [30], Murugusundaramoorthy and Magesh [17] and applying coefficient inequality (12) of Lemma 3.1, we estimate certain inclusion relations involving the classes $k$-$\mathcal{UCV}$, $k$-$\mathcal{ST}$ and $\mathcal{G}_\nu^\delta(\lambda,\beta)$ if some parametric inequalities hold.

**Theorem 4.1.** *Let $\nu > -2, \delta > 3$. If $f \in \mathcal{S}$ and if the parametric inequality*

$$\frac{\Gamma(\nu+2+\delta)\Gamma(\delta-2)}{\Gamma(\delta)\Gamma(\nu+\delta)}\left[1-\beta+(\nu+2)(\delta-2)[6\lambda(\nu+3)(\delta-3)+2(2\lambda+1)]\right] \leq 2(1-\beta) \tag{25}$$

*holds, then $\widetilde{\mathcal{I}}_\nu^\delta : \mathcal{S}^* \to \mathcal{G}_\nu^\delta(\lambda,\beta)$.*

*Proof.* Let $f$ of the form (1) belong to the class $\mathcal{S}^*$, then we get $|a_n| \leq n, n \geq 2$ to show $\widetilde{\mathcal{I}}_\nu^\delta \in \mathcal{G}_\nu^\delta(\lambda,\beta)$ by the coefficient inequality (12), we need to show that

$$\mathcal{G}(\delta,\nu,\lambda,\beta) = \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\Theta_n(\delta,\nu)|a_n| \leq 1-\beta \tag{26}$$

where $\Theta_n(\delta,\nu)$ is given by (9). Hence, we deduce that

$$\begin{aligned}\mathcal{G}(\delta,\nu,\lambda,\beta) &= \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}|a_n| \\ &\leq \sum_{n=2}^{\infty}n(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} \\ &= \sum_{n=2}^{\infty}[n^3\lambda+(1-\lambda)n^2-n\beta]\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}.\end{aligned} \tag{27}$$

Writing

$$n^3 = n(n-1)(n-2)+3n(n-1)+n, \quad n^2 = n(n-1)+n, \tag{28}$$

from (27), we have,

$$\begin{aligned}\mathcal{G}(\delta,\nu,\lambda,\beta) \leq{}& \lambda\sum_{n=2}^{\infty}n(n-1)(n-2)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} \\ &+(2\lambda+1)\sum_{n=2}^{\infty}n(n-1)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}+(1-\beta)\sum_{n=2}^{\infty}n\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}.\end{aligned}$$

Now by using the equations (22) to (24) we get

$$\begin{aligned}\mathcal{G}(\delta,\nu,\lambda,\beta) \leq{}& \lambda\frac{6(\nu+2)(\nu+3)\Gamma(\nu+2+\delta)\Gamma(\delta-4)}{\Gamma(\delta)\Gamma(\nu+\delta)} \\ &+(2\lambda+1)\frac{2(\nu+2)\Gamma(\nu+2+\delta)\Gamma(\delta-3)}{\Gamma(\delta)\Gamma(\nu+\delta)}+(1-\beta)\left(\frac{\Gamma(\nu+2+\delta)\Gamma(\delta-2)}{\Gamma(\delta)\Gamma(\nu+\delta)}-1\right).\end{aligned}$$

The last expression is bounded above by $1-\beta$ if and only if (25) is satisfied. Hence the proof of Theorem 4.1 is completed. □

**Theorem 4.2.** *Let $\nu > -2, \delta > 2$. If $f \in \mathcal{S}$ and if the parametric inequality*

$$\frac{\Gamma(\nu+2+\delta)\Gamma(\delta-1)}{\Gamma(\delta)\Gamma(\nu+\delta)}\left((\delta-1)\{1+2\lambda(\nu+2)(\delta-2)\}-\frac{\beta}{\nu+\delta}\right) \leq 2(1-\beta) \tag{29}$$

*holds, then $\widetilde{\mathcal{I}}_\nu^\delta : \mathcal{K} \to \mathcal{G}_\nu^\delta(\lambda,\beta)$.*

*Proof.* Let $f$ of the form (1) belong to the class $\mathcal{K}$, then we get $|a_n| \leq 1, n \geq 2$ to show $\widetilde{\mathcal{I}}_\nu^\delta \in \mathcal{G}_\nu^\delta(\lambda,\beta)$ by the coefficient inequality (12), we need to show that

$$\mathcal{G}(\delta,\nu,\lambda,\beta) = \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\Theta_n(\delta,\nu)|a_n| \leq 1-\beta \tag{30}$$

where $\Theta_n(\delta,\nu)$ is given by (9). Hence,

$$\mathcal{G}(\delta,\nu,\lambda,\beta) \leq \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}.$$

Thus, we have

$$\mathcal{G}(\delta,\nu,\lambda,\beta) \leq \lambda\sum_{n=2}^{\infty} n(n-1)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} + \sum_{n=2}^{\infty} n\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}} - \beta\sum_{n=2}^{\infty}\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}.$$

Using the equations (21) to (23) we get

$$\begin{aligned}\mathcal{G}(\delta,\nu,\lambda,\beta) \leq{}& \lambda\frac{2(\nu+2)\Gamma(\nu+2+\delta)\Gamma(\delta-3)}{\Gamma(\delta)\Gamma(\nu+\delta)} + \left(\frac{\Gamma(\nu+2+\delta)\Gamma(\delta-2)}{\Gamma(\delta)\Gamma(\nu+\delta)} - 1\right)\\ &- \beta\left(\frac{\Gamma(\nu+2+\delta)\Gamma(\delta-1)}{\Gamma(\delta)\Gamma(\nu+1+\delta)} - 1\right).\end{aligned}$$

The last expression is bounded above by $1-\beta$ if and only if (29) is satisfied. Hence the proof is completed. ☐

**Theorem 4.3.** *Let $\nu > -2, \delta > 2$ and $P_1 = P_1(k)$ given by (3). If $f \in k$-$\mathcal{UCV}$, for some $k$ $(0 \leq k < \infty)$ and if the parametric inequality*

$$\begin{aligned}&\frac{\Gamma(\nu+2+\delta)}{\Gamma(\delta)\Gamma(\nu+2+\delta-P_1)}\left(\Gamma(\delta-P_1)+\lambda(\nu+2)P_1\Gamma(\delta-P_1-1)\right)\\ &-\frac{\Gamma(\nu+2+\delta)}{\Gamma(\delta)\Gamma(\nu+2+\delta-P_1)}\frac{\beta}{(\nu+1)(P_1-1)}\left(\Gamma(\delta-P_1+1)-\frac{\Gamma(\delta)\Gamma(\nu+2+\delta-P_1)}{\Gamma(\nu+1+\delta)}\right)\\ &\leq 2(1-\beta)\end{aligned} \tag{31}$$

*is satisfied, then $\widetilde{\mathcal{I}}_\nu^\delta f \in \mathcal{G}_\nu^\delta(\lambda,\beta)$.*

*Proof.* Let $f$ be of the form (1) belong $k$-$\mathcal{UCV}$. By using (12), we need to show that $\widetilde{\mathcal{I}}_\nu^\delta f \in \mathcal{G}_\nu^\delta(\lambda,\beta)$, it is sufficient to prove that

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}|a_n| \leq 1-\beta.$$

Let

$$\mathcal{P}(\delta,\nu,\lambda,\beta) = \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}|a_n|.$$

Now by substituting for $|a_n|$ given by (4) and proceeding as in the proof of Theorem 4.1, we get

$$\begin{aligned}
\mathcal{P}(\delta,\nu,\lambda,\beta) &= \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}|a_n| \\
&\leq \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_n} \\
&\leq \sum_{n=2}^{\infty}\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_{n-1}}+\lambda\sum_{n=2}^{\infty}\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_{n-2}} \\
&- \beta\sum_{n=2}^{\infty}\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_n} \\
&\leq \left(\frac{\Gamma(\nu+2+\delta)\Gamma(\delta-P_1)}{\Gamma(\delta)\Gamma(\nu+2+\delta-P_1)}-1\right) \\
&+ \lambda\frac{(\nu+2)P_1}{(\nu+2+\delta)}\sum_{n=2}^{\infty}\frac{(\nu+3)_{n-2}}{(\nu+3+\delta)_{n-2}}\frac{(1+P_1(k))_{n-2}}{(1)_{n-2}} \\
&- \beta\frac{(\nu+1+\delta)}{(\nu+1)(P_1-1)}\sum_{n=2}^{\infty}\frac{(\nu+1)_n}{(\nu+1+\delta)_n}\frac{(P_1-1)_n}{(1)_n} \\
&= \left(\frac{\Gamma(\nu+2+\delta)\Gamma(\delta-P_1)}{\Gamma(\delta)\Gamma(\nu+2+\delta-P_1)}-1\right) \\
&+ \lambda\frac{(\nu+2)P_1}{(\nu+2+\delta)}\left(\frac{\Gamma(\nu+3+\delta)\Gamma(\delta-P_1-1)}{\Gamma(\delta)\Gamma(\nu+2+\delta-P_1)}\right) \\
&- \beta\frac{(\nu+1+\delta)}{(\nu+1)(P_1-1)}\left(\frac{\Gamma(\nu+1+\delta)\Gamma(\delta-P_1+1)}{\Gamma(\delta)\Gamma(\nu+2+\delta-P_1)}-1\right)+\beta.
\end{aligned}$$

The last expression is bounded above by $1-\beta$ if and only if (31) is satisfied. Hence the proof is completed. $\square$

**Theorem 4.4.** *Let $\nu > -2, \delta > 2$ and $P_1 = P_1(k)$ given by (3). If $f \in k$-$\mathcal{ST}$, for some $k$ $(0 \leq k < \infty)$ and if the parametric inequality*

$$\begin{aligned}
{}_3F_2(\nu+2,P_1,2;\nu+2+\delta;1;1)+2\lambda\frac{(\nu+2)P_1}{(\nu+2+\delta)}\ {}_3F_2(\nu+3,1+P_1,3;\nu+3+\delta,2;1) \\
-\beta\ {}_2F_1(\nu+2,P_1;\nu+2+\delta;1)\leq 2(1-\beta) \qquad (32)
\end{aligned}$$

*is satisfied, then $\widetilde{\mathcal{I}}_{\nu}^{\delta}f \in \mathcal{G}(\lambda,\beta)$.*

*Proof.* Let $f$ be given by (1) belong to $k$-$\mathcal{ST}$. By (12)we have

$$\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}|a_n| \leq 1-\beta.$$

Substituting for $|a_n|$ given by (5), we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}|a_n| \\
&\le \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\beta)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_{n-1}} \\
&\le \lambda\sum_{n=2}^{\infty}n(n-1)\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_{n-1}}+\sum_{n=2}^{\infty}n\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_{n-1}} \\
&\qquad -\beta\sum_{n=2}^{\infty}\frac{(\nu+2)_{n-1}}{(\nu+2+\delta)_{n-1}}\frac{(P_1(k))_{n-1}}{(1)_{n-1}} \\
&= [\ {}_3F_2(\nu+2,P_1,2;\nu+2+\delta;1;1)-1] \\
&\qquad +2\lambda\frac{(\nu+2)P_1}{(\nu+2+\delta)}\ {}_3F_2(\nu+3,1+P_1,3;\nu+3+\delta,2;1) \\
&\qquad -\beta\ [\ {}_2F_1(\nu+2,P_1;\nu+2+\delta;1)-1].
\end{aligned}$$

The last expression is bounded above by $1-\beta$ if and only if (32) is satisfied. Hence the proof is completed. □

**Concluding Remarks:** For the different choices of $\delta$ and $\nu$,it is of interest to note that $\delta=-\nu=\lambda>-1$, $\widetilde{\mathcal{I}}^{\lambda}_{-\lambda}f(z)\equiv\mathcal{D}^{\lambda}f(z)$ the Ruscheweyh derivative operator [25] and $\widetilde{\mathcal{I}}^{c-a}_{a-2}f(z)\equiv\mathcal{L}(a,c)$ is the Carlson-Shaffer operator [3], hence one can deduce analogous results given in Theorems 4.1 to 4.4 for the function class defined in this paper involving Ruscheweyh and Carlson-Shaffer derivative operators and we omit the details involved.

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## References

[1] Bailey, W.N., (1935), Generalised Hypergeometric Series, Cambridge, England: Cambridge University Press, pp.2-3.

[2] de Branges, L., (1985), A proof of the Bierberbach conjucture, Acta. Math., 154, pp.137–152.

[3] Carlson, B.C., Shaffer, D.B., (1984), Starlike and prestarlike hypergeometric functions, SIAM. J. Math. Anal., 15, pp.737–745.

[4] Dziok, J., Murugusundaramoorthy, G., (2010), A generalized class of starlike functions associatedwith the wright hypergeometric function, Mathematica Veslnik, 62, 4, pp. 271–283.

[5] Goodman, A.W., (1991), On uniformly convex functions, Ann. Polon. Math., 56, pp.87–92.

[6] Goodman, A.W., (1991), On uniformly starlike functions, J. Math. Anal. Appl., 155(2), pp.364–370.

[7] Hohlov, Y.E., (1978), Operators and operations in the class of univalent functions, Izv. Vysš.Učebn. Zaved. Mathmatika, 10, pp.83–89 (in Russian).

[8] Kanas, S., Winiowska, A., (1998), Conic Regions and $k$-uniform convexity-II, Folia. Sci.Univ. Tech. Resoviensis Mat., 22(170), pp.65–78.

[9] Kanas, S., (2005), Coefficient estimates in subclasses of the Caratheodory Class Related to Conical Domains, Acta Math. Univ. Comeniane, LXXIV(2), pp.149–161.

[10] Kanas, S., Winiowska, A., (1999), Conic regions and $k$-uniform convexity, J. Comput. Appl. Math., 105, pp. 327–336.

[11] Kanas, S., Srivastava, H. M., (2000), Linear operators associated with k-uniformly convex functions, Integral Transforms Spec. Funct., 9(2), pp.121– 132.

[12] Kanas, S., Winiowska, A., (2000), Conic regions and $k$-starlike functions, Rev. Roumaine Math. Pures Appl., 45(4), pp.647–657.

[13] Keogh, F.R., Merkes, E.P., (1969), A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20, pp.8–12.

[14] Kim, Y.C., Srivastava, H.M., (1997), Fractional integral and other linear operators associated with Gaussian hypergeometric function, Complex Var. Theory Appl., 34, pp.293–312

[15] Merkes, E., Scott, B.T., (1961), Starlike Hypergeometric functions, Proc. Amer. Math. Soc., 12, pp.885–888.

[16] Mishra, A.K., Gochhayat, P., (2010), Fekete-Szego problem for a class defined by an integral operator, Kodai Math. J., 33, pp.310–328.

[17] Murugusundaramoorthy, G., Magesh, N., (2011), On Certain subclasses of analytic functions associated with hypergeometric functions, Applied Mathematics Letters, 24, pp.494–500.

[18] Nunokawa, M., Owa, S., Obradovic, M., Aouf, M.K., Saitoh, H., Ikeda, A., Koike, N., (1996), Sufficient conditions for starlikeness, Chinese Journal of Mathematics, 24(3), pp. 265–271.

[19] Obradovic, M., Joshi, S.B., (1998) On Certain classes of strongly starlike functions, Taiwanese J. Math., 2(3), pp.297–302.

[20] Owa, S., Srivastava, H.M., (1987), Univalent and starlike generalised hypergeometric functions, Canad. J. Math., 39, pp. 1057–1077.

[21] Padmanabhan, K.S., (2001), On sufficient conditions for starlikeness, Indian J. Pure Appl. Math., 32, pp.543–550.

[22] Ramesha, C., Sampath Kumar, Padmanabhan, K.S., (1995), A sufficient condition for starlikeness, Chinese Journal of Mathematics, 23(2), pp.167–171.

[23] Ronning, F., (1991), On starlike functions associated with parabolic regions, Ann. Univ. Marie Curie-Sklodowska Sect., A 45, pp.117–122.

[24] Ruscheweyh, S., Singh, V., (1986), On the order of starlikeness of hypergeometric functions, J. Math. Anal. Appl., 113, pp.1–11.

[25] Ruscheweyh, S., (1975), New criteria for univalent functions, Proc. Amer. Math. Soc., 49, pp.109–115.

[26] Robertson, M.S., (1936), On the theory of univalent functions, Ann. Math., 37, pp.374–408.

[27] Sharma, P., Jain, V., (2013), A Generalized class of k-starlike functions involving a calculus operator, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 83(3), pp.247–252.

[28] Silverman, H., (1975), Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51, pp.109–116.

[29] Silverman, H., (1993), Starlike and convexity properties for hypergeometric functions, J. Math. Anal. Appl., 172(3), pp.574–581.

[30] Srivastava, H.M., Murugusundaramoorthy, G., Sivasubramanian, S., (2007), Hypergeometric functions in the parabolic starlike and uniformly convex domains, Integral Transform Spec. Funct., 18, pp.511–520.

[31] Srivastava, H.M., Mishra, A.K., (2000), Applications of fractional calculus to parabolic starlike and uniformly convex functions, Comput. Math. Appl., 39(3/4), pp.57–69.


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